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THE OPTIMAL DAMPER OF THE LONGITUDINAL OSCILLATIONS OF A ROD†

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Taking the example of the longitudinal oscillation of a rod, the parameters of an end damper are found which removes any perturbations in the system in the least time, equal to twice the time required for a wave to pass along the rod and is optimal in this sense. © 1997 Elsevier Science Ltd. All rights reserved.

The problem of determining an optimal dynamic oscillation damper was apparently formulated for the first time by Timoshenko [1] as applied to a system in the form of a point mass fastened to a base by means of a spring. An elastic-inertial device with a viscous damper is connected to the mass which is excited by means of a periodic force.

A rigorous and complete solution of the optimization problem was obtained in [2] for this simplest of systems, and the damping parameters were determined for which the rate of damping (the decrement) of the free oscillations of the system is the maximum possible. The results were extended to the case of the damping of the oscillations of a system with two degrees of freedom. Investigations were carried out in [2] from the position of the theory of the oscillations of linear lumped-parameter systems for the analogous problem in the case of a chain of coupled oscillators which simulate the longitudinal oscillations of a rod and, naturally, a solution was sought in a restricted class of possible situations which exclude absolute damping in a finite time.

The problem of finding the damping parameters which ensure damping in a finite time can be formulated and solved within the framework of distributed systems by analysing dynamic processes using wave theory. The possibility of creating a damper which produces the fastest damping in distributed systems, was indicated for the first time in [3].

We consider a rod which is rigidly clamped at one end ($x = 0$) and is secured at the other end ($x = l$) by means of a viscous damper (Fig. 1).

The propagation of longitudinal perturbations in the rod is described by the initial-boundary-value problem

$$u_{tt} - c^2 u_{xx} = 0 \tag{1}$$

$$u|_{x=0} = 0; \quad EFu_x + \alpha u_t|_{x=l} = 0 \tag{2}$$

$$u(x,0) = U_0(x), \quad u_t(x,0) = V_0(x) \quad (0 \leqslant x \leqslant l) \tag{3}$$

Here, $u(x, t)$ is the longitudinal displacement of the cross-section of the rod relative to the unperturbed state, $c = (E/\rho)^{1/2}$ is the velocity of the longitudinal waves, E and ρ are the modulus of elasticity and the density of the rod material, F is the cross-sectional area of the rod, α is the coefficient of viscous drag, and $U_0(x)$ and $V_0(x)$ are specified functions.

The solution of wave equation (1) can be represented in the form of two waves travelling in opposite directions

$$u(x,t) = f(t + x/c) + h(t - x/c) \tag{4}$$

The expansion of the required solution in travelling waves enables one to change from a partial differential boundary-value problem to an ordinary differential equation with a divergent parameter which describes the change in the form of the wave $f(t) = -h(t)$ when it interacts with the boundary $x = l$

$$EFc^{-1}[f'(t+\tau) + f'(t-\tau)] + \alpha[f'(t+\tau) - f'(t-\tau)] = 0 \tag{5}$$

where $\tau = l/c$ is the time the wave takes to travel along the rod.

In order to solve the problem, initial conditions are required for $f'(t \pm x/c)$. These are found by expanding the initial perturbations (3) in travelling waves when $t = 0$

$$f'(\pm x/c) = \tfrac{1}{2}[\pm V_0(x) + cU_0'(x)], \quad 0 \leqslant x \leqslant l$$

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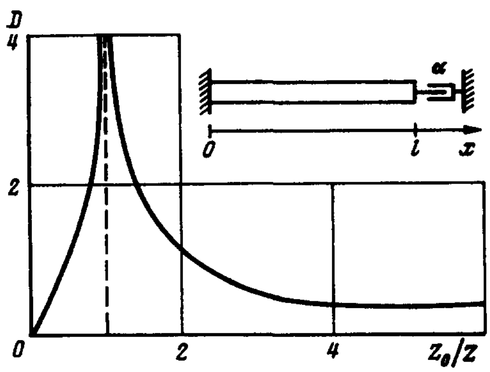

Fig. 1.

The boundary $x = 0$ is absolutely rigid ($u(0, t) = 0$) and therefore, by continuing the functions $V_0(x)$ and $U_0(x)$ in the interval $[-l, 0]$ in an even and odd manner respectively, we obtain the initial conditions for f' in the form

$$f'(t) = \tfrac{1}{2}[V_0(ct) + cU_0'(ct)], \quad f'(-t) = -f'(t) \quad (0 \leq t \leq \tau)$$

On putting $f'(t) = \varphi(t)$, we find that the initial problem reduces to one of finding the solution of the difference equation

$$\varphi(t+\tau) - \gamma\varphi(t-\tau) = 0, \quad \gamma = (\alpha - EF/c)/(\alpha + EF/c) \tag{6}$$

which satisfies the condition

$$\varphi(t) = \tfrac{1}{2}[V_0(ct) + cU_0'(ct)], \quad \varphi(-t) = -\varphi(t), \quad (0 \leq t \leq \tau) \tag{7}$$

We will seek a solution of Eq. (6) in the form [4]

$$\varphi(t) = \Pi(t)\exp(-\beta t) \tag{8}$$

where $\Pi(t)$ is a certain periodic function with period 2τ and β is a constant which depends on the parameters of the system.

On substituting (8) into (6), we find that the quantity $\beta = \ln \gamma^{-1}/(2\tau)$, that is, the solution (8) is an infinite set of 2τ-periodic functions which differ in their factors $\exp(ik\pi/\tau)$ ($k = 0, \pm 1, \pm 2, \ldots$). Henceforth, we shall therefore understand $\varphi(t)$ to be the "principal value" for which $k = 0$ and ln denotes the principal value of the logarithm.

Hence, the solution (8) can be written in the form

$$\varphi(t) = \Pi(t)\exp(t\ln\gamma/(2\tau)) \text{ when } \gamma > 0 \quad (\alpha > EF/c) \tag{9}$$
$$\varphi(t) = \Pi(t)\exp[(\ln(-\gamma) + i\pi)t/(2\tau)] \text{ when } \gamma < 0 \quad (\alpha < EF/c)$$

The function $\Pi(t)$ in the interval $[-\tau, \tau]$ which occurs in (8) is defined by the initial conditions

$$\Pi(t) = \varphi(t)\exp(\beta t)$$

and $\varphi(t)$ is found from relation (7).

We continue $\Pi(t)$ periodically outside this interval

$$\Pi(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k^+ \cos\frac{\pi k t}{\tau} + a_k^- \sin\frac{\pi k t}{\tau}\right)$$

$$a_k^+ = \frac{1}{\tau}\int_{-\tau}^{\tau}\Pi(\theta)\cos\frac{\pi k\theta}{\tau}d\theta, \quad a_k^- = \frac{1}{\tau}\int_{-\tau}^{\tau}\Pi(\theta)\sin\frac{\pi k\theta}{\tau}d\theta$$

Hence, if the functions of the initial perturbations U_0 and V_0 are known, the required solution $u(x, t)$ is determined in terms of the function $\varphi(t)$ by integration

$$u(x,t) = \int_{t-x/c}^{t+x/c}\varphi(\theta)d\theta$$

After some reduction, we obtain the solution of problem (1)–(3) when $\gamma > 0$ ($\alpha > EF/c$)

$$u(x,t) = f(t + x/c) - f(t - x/c) \tag{10}$$

$$f(x) = \frac{1}{\ln\gamma}\exp\left(\frac{x}{2\tau}\ln\gamma\right)\left[a_0 + \sum_{k=1}^{\infty}\left(p_k^+\cos\frac{\pi k x}{\tau} + p_k^-\sin\frac{\pi k x}{\tau}\right)\right]$$

$$p_k^{\pm} = \frac{2\tau(a_k^{\pm}\ln\gamma \mp 2\pi k a_k^{\mp})\ln\gamma}{\ln^2\gamma + (2\pi k)^2}$$

When $\gamma < 0$ ($\alpha < EF/c$), after separating the real and imaginary parts, the solution has the form (10), where

$$f(x) = \exp\left[\frac{x}{2\tau}\ln(-\gamma)\right]\sum_{k=-\infty}^{\infty}\left(q_k^+\cos\frac{\pi + 2\pi k}{2\tau}x + q_k^-\sin\frac{\pi + 2\pi k}{2\tau}x\right)$$

$$q_0^+ = \frac{a_0\tau\ln(-\gamma)}{\pi^2 + \ln^2(-\gamma)}, \quad q_0^- = \frac{a_0\tau\pi}{\pi^2 + \ln^2(-\gamma)} \tag{11}$$

$$q_k^{\pm} = \frac{a_k^{\pm}\ln(-\gamma) \mp a_k^{\mp}(\pi + 2\pi k)}{\ln^2(-\gamma) + (\pi + 2\pi k)^2}\tau$$

In particular, the displacement of the rod at its boundary $x = l$ is equal to

$$u(l,t) = \ln\gamma^{-1}2\exp\frac{t\ln\gamma}{2\tau}\operatorname{sh}\frac{\ln\gamma}{2}\left\{a_0 + \sum_{k=1}^{\infty}(-1)^k\left[p_k^+\cos\frac{\pi k t}{\tau} + p_k^-\sin\frac{\pi k t}{\tau}\right]\right\}, \quad \gamma > 0$$

$$u(l,t) = 2\exp\frac{t\ln(-\gamma)}{2\tau}\operatorname{ch}\frac{\ln(-\gamma)}{2}\left\{\sum_{k=-\infty}^{\infty}(-1)^k\left[q_k^-\cos\frac{\pi + 2\pi k}{2\tau}t - q_k^+\sin\frac{\pi + 2\pi k}{2\tau}t\right]\right\}, \quad \gamma < 0 \tag{12}$$

Expressions (10)–(12) give, in general form, the solution of the problem of the damping of the longitudinal oscillations of a rod which is rigidly clamped at one end.

Solution (12) shows that, at the initial stage of the motion when the time does not exceed twice the time taken for the wave to travel along the rod ($t < 2\tau$), initial conditions exist under which an increase in the amplitude of the longitudinal displacements is possible in this interval. When $t > 2\tau$, the longitudinal displacements of the rod in each cross-section decay exponentially: $u(x, t) \sim \exp(t \ln|\gamma|/(2\tau))$.

The rate of decay of the oscillations is characterized by the quantity $D = -\ln|\gamma|$ which has the meaning of the logarithmic decrement of the oscillations.

By analogy with electrodynamics and acoustics, we shall call $EF/c = Z_0$ the impedance of the rod for longitudinal waves and $\alpha = Z$ the damping impedance. The dependence of the logarithmic decrement $D = \ln|(1 + Z_0/Z)/(1 - Z_0/Z)$ on a quantity which is equal to the ratio of the impedance of the rod to the damping impedance Z_0/Z is shown in Fig. 1.

It is seen that, when $Z = Z_0$ and the impedance of the rod and the damping impedance are equal, the logarithmic decrement tends to infinity, that is, absolute damping of the oscillations for any initial perturbations starting from a time $t = 2\tau$ occurs practically instantaneously. Such a damping provides the fastest damping of the oscillations in the system and it is optimal in this sense.

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